On the Hausdorff dimension faithfulness of Vitaly coverings and its applications

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The talk is devoted to some new results in the theory of the Hausdorff dimension calculation (faithfulness of Vitaly coverings for the determination of the Hausdorff dimension, transformations preserving the Hausdorff dimension) and to some applications of these results (metric and dimensional number theory, theory of singularly continuous probability measures).

It is well known that in many situations the determination (or even estimations) of the Hausdorff dimension for sets from a given family or even for a given set is a rather non-trivial problem. To simplify the calculation of this dimension it is extremely useful to have an appropriate and a relatively narrow family of admissible coverings which lead to the same value of the dimension.

Let Φ be a Vitaly covering on $[0,1]$. Let us recall that *the α-dimensional Hausdorff measure* of a set $E \subset [0,1]$ w.r.t. a given Φ is defined by

$$H^{\alpha}(E,\Phi) = \lim_{\varepsilon \to 0} \left[\inf_{|E_j| \leq \varepsilon} \left\{ \sum_j |E_j|^{\alpha} \right\} \right] = \lim_{\varepsilon \to 0} H^{\alpha}_{\varepsilon}(E,\,\Phi),$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of E, $E_j \in \Phi$.

The nonnegative number

$$\dim_H(E,\,\Phi) = \inf\{\alpha:\;\; H^{\alpha}(E,\,\Phi) = 0\}$$

is called the Hausdorff dimension of the set $E \subset [0,1]$ w.r.t. a family Φ.

Definition. A covering family Φ is said to be *faithful family of coverings (non-faithful family of coverings)* for the Hausdorff dimension calculation on $[0,1]$ if

$$\dim_H(E,\Phi) = \dim_H(E), \quad \forall E \subseteq [0,1]$$

$$(\text{resp. } \;\exists E \subseteq [0,1] : \dim_H(E,\Phi) \neq \dim_H(E)).$$

It is clear that any family Φ of comparable net-coverings (i.e., net-coverings which generate comparable net-measures) is faithful. Conditions for a covering family to be faithful were studied by many authors. The family of cylinders of the classical continued fraction expansion can probably be considered as the first (and rather unexpected) example of non-faithful one-dimensional net-family of coverings. By using approach, which has been invented by Yuval Peres to prove non-faithfulness of the family of continued fraction cylinders, in S. Albeverio, Yu. Kondratiev, R. Nikiforov and G.Torbin have proven the non-faithfulness for the family of cylinders of Q_∞-expansion with polynomially decreasing elements $\{q_i\}$. Necessary and sufficient conditions for the family of cylinders generated by the Cantor series expansion were found recently by S. Albeverio, G. Ivanenko, M. Lebid and G.Torbin (to the best of our knowledge this gives the first necessary and sufficient condition of the faithfulness for a class of covering families containing both faithful and non-faithful ones). During the talk we present some new general necessary and sufficient conditions for a covering family to be faithful and new techniques for proving faithfulness/non-faithfulness for the family of cylinders generated by different expansions of real numbers. In particular, necessary and sufficient conditions for the Hausdorff dimension faithfulness of the family of cylinders, generated by GLS-expansions will be presented. Connections between faithfulness of net coverings and the theory of DP-transformations (i.e., transformations preserving the Hausdorff dimension) will also be discussed.

Applications for the dimensional theory of non-normal numbers and for the theory of singularly continuous probability measures will also be discussed.

The talk is based on joint research with S. Albeverio, Yu. Kondratiev, M. Lupain, R. Nikiforov and O. Smiyan.

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